Linear Algebra Lecture Notes

Mohammad Saleh, Email:msaleh@birzeit.edu
Mathematics Department, Birzeit University, West Bank, Palestine

These lecture notes is not an alternative for the class lectures

Chapter 1

Linear system of equations and matrices

1.1 Systems of Equations

Systems of equations are either

1. Linear system: If all equations in the system are linear

   or

2. Nonlinear system: At least one of the equations in the system are nonlinear

Example 1.1.1. *:* $\begin{array}{rcrcl} x & - & y & = & 1 \\ 2x & + & 3y & = & 2 \end{array}$ *is linear*

$\begin{array}{rcrcl} x & - & y & = & 1 \\ 2x^2 & + & 3y & = & 2 \end{array}$ *is nonlinear linear*

In this course we study only linear systems.

1.2 A general form of the linear system:

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + ... + a_{3n}x_n = b_3 \\ . \\ . \\ . \\ . \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{array}$$

, where a_{ij}, b_i are all real numbers, is called an $m \times n$ linear system

Definition 1.2.1. *A solution of the above system is a set of real numbers* $c_1, c_2, ..., c_n$ *such that if substitute* $x_i = c_i$ *then all equations in the above system holds denoted by* $(c_1, c_2, ..., c_n)$ *or* $\begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_n \end{pmatrix}$.

Example 1.2.1. $\begin{array}{ccccc} x & - & y & = & 1 \\ x & + & 3y & = & 5 \end{array}$ *has a solution* $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Definition 1.2.2. *A linear system is called a* ***square*** *system if* $m = n$ *and it is called an* $n \times n$ *linear system*

Definition 1.2.3. *A linear system is called* ***consistent*** *if it has a solution, and it is called* ***inconsistent*** *if it has no solution*

Consistent linear systems has either a unique solution or infinite number of solutions

A 2×2 linear system :

Example 1.2.2. $\begin{array}{ccccc} x & - & y & = & 1 \\ x & + & 3y & = & 5 \end{array}$ *has a unique solution* $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Example 1.2.3. $\begin{array}{ccccc} x & - & y & = & 1 \\ 2x & - & 2y & = & 2 \end{array}$ *has infinite number of solutions*

Example 1.2.4. $\begin{array}{rcrcl} x & - & y & = & 1 \\ 2x & - & 2y & = & 5 \end{array}$ *has no solution*

Definition 1.2.4. ***Equivalent systems:*** *Two linear systems are called equivalent systems if they have the same variables(unknowns) and the same solution set.*

Operations on the linear systems:

1. Interchange two equations
2. Multiply an equation by a nonzero constant
3. add a number to both sides of an equation (add a multiple of an equation to another equation)

A strict upper triangular system:

1. It is a square system
2. $a_{k1} = a_{k2} = .. = a_{k,k-1} = 0$ in the kth equation (in the $k-$th equation, the coefficients of the first $k-1$ variables are zeros)
3. $a_{kk} \neq 0$ (the coefficient of x_k is nonzero)

Which is of the form

$$\begin{array}{cccccc} a_{11}x_1 & +a_{12}x_2 & +...... & & +a_{1n}x_n & = b_1 \\ & a_{22}x_2 & +..... & & +a_{2n}x_n = & b_2 \\ & & a_{33}x_3 & +..... & +a_{3n}x_n = & b_3 \\ & & . & & & \\ & & . & & & \\ & & . & & & \\ & & & & a_{nn}x_n & = b_n \end{array}$$

A strict upper triangular system has a unique solution and we solve it by backward substitution.

Example 1.2.5. $\begin{array}{cccc} x_1 & -x_2 & +x_3 & = 2 \\ & 2x_2 & -x_3 = & 1 \\ & & 2x_3 & = 2 \end{array}$

has a solution $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

1.3 1.2. Row Echelon form and solutions of linear systems

Definition 1.3.1. *A matrix is an array of numbers or objects arranged in rows and columns denoted by $A, B, C, ...$*

A matrix A with m rows and n columns is called an $m \times n$ matrix read m by n matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ . & & & \\ . & & & \\ . & & & \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}$$

The entry of a matrix A in the $i-$th row and $j-$th column is called the $ij-$th entry denoted by a_{ij}

Augmented matrix of a linear system

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + ... + a_{3n}x_n = b_3 \\ . \\ . \\ . \\ . \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{array}$$

Definition 1.3.2. ***The Augmented matrix*** *of a linear system above denoted by* $\overline{\mathbf{A}} = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} & |b_1 \\ a_{21} & a_{22} & ... & a_{2n} & |b_2 \\ . & & & & \\ . & & & & \\ . & & & & \\ a_{m1} & a_{m2} & ... & a_{mn} & |b_m \end{pmatrix}$

Elementary row operations:

1. Interchange two rows
2. Multiply a row by a nonzero constant

3. Replace a row by its sum with a multiple of another row (add a multiple of a one row to another row)

Row Echelon Form (REF)

Example 1.3.1. : $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ *is in REF*

$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ *is not in REF*

$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ *is not in REF*

$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ *is not in REF*

$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *is in REF*

Definition 1.3.3. *An $m \times n$ matrix is in REF iff:*

1. *The first nonzero entry in a nonzero row is 1 called the leading one or the pivot 1*
2. *the leading one in the $k-th$ row is to the right of the leading one in the $k-1$-row*
3. *Zero rows are below the nonzero rows*

Remark 1.3.4. Any matrix can be written in REF using the row operations

Gauss Elimination Method is a method to solve linear systems by using row operations on the augmented matrix $\overline{A}$ of the system to change it in REF

Example 1.3.2. . *Use Gauss Elimination method to Solve*

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 2 \\ x_1 + 2x_2 - x_3 &= 1 \\ -x_1 + x_2 - 2x_3 &= -2 \end{aligned}$$

Solution $\overline{A} = \begin{pmatrix} 1 & -1 & 3 & |2 \\ 1 & 2 & -1 & |1 \\ -1 & 1 & -2 & |-2 \end{pmatrix} R_2 - R_1, R_3 + R_1 \rightarrow \begin{pmatrix} 1 & -1 & 3 & |2 \\ 0 & 3 & -4 & |-1 \\ 0 & 0 & 1 & |0 \end{pmatrix}$

$\frac{1}{3}R_2 \rightarrow \begin{pmatrix} 1 & -1 & 3 & |2 \\ 0 & 1 & \frac{-4}{3} & |\frac{-1}{3} \\ 0 & 0 & 1 & |0 \end{pmatrix}$

$x_3 = 0, x_2 = \frac{-1}{3}, x_1 = \frac{5}{3}$

Solution $\begin{pmatrix} \frac{5}{3} \\ \frac{-1}{3} \\ 0 \end{pmatrix}$.

Example 1.3.3. . *Use Gauss Elimination method to Solve*

$x_1 - x_2 + 3x_3 = 2$

$x_1 + 2x_2 - x_3 = 1$

Solution $\overline{A} = \begin{pmatrix} 1 & -1 & 3 & |2 \\ 1 & 2 & -1 & |1 \end{pmatrix} R_2 - R_1 \rightarrow \begin{pmatrix} 1 & -1 & 3 & |2 \\ 0 & 3 & -4 & |-1 \end{pmatrix}$

$\frac{1}{3}R_2 \rightarrow \begin{pmatrix} 1 & -1 & 3 & |2 \\ 0 & 1 & \frac{-4}{3} & |\frac{-1}{3} \end{pmatrix}$

x_3 *is free* , x_1, x_2 *leading* , *so let* $x_3 = \alpha \in R$, *then form equation2,* $x_2 = \frac{-1}{3} + \frac{4}{3}\alpha$, *and from equation1,* $x_1 = 2 + x_2 - 3x_3 = 2 + \frac{-1}{3} + \frac{4}{3}\alpha - 3\alpha = \frac{5}{3} - \frac{5}{3}\alpha$

solution $\begin{pmatrix} \frac{5}{3} - \frac{5}{3}\alpha \\ \frac{-1}{3} + \frac{4}{3}\alpha \\ \alpha \end{pmatrix}$.

Remark 1.3.5. If we have more than one linear systems of the form $(A|b_1), (A|b_2), ..., (A|b_k)$, then we can solve the systems simultaneously by forming the augmented matrix $(A|b_1|b_2|, ..., |b_k)$

Reduced Row Echelon Form (RREF) An $m \times n$ **matrix is in RREF** iff:

1. It is in REF
2. The leading 1 is the only nonzero in that column

Example 1.3.4. : $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *is in RREF*

$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *is not in RREF*

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \textit{ is not in RREF}$$
$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \textit{ is not in RREF}$$
$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \textit{ is in RREF}$$

Remark 1.3.6. Any matrix can be written in RREF using the row operations

Gauss-Jordan Elimination Method is a method to solve linear systems by using row operations on the augmented matrix $\overline{A}$ of the system to change it in RREF

Example 1.3.5. . *Use Gauss Elimination-Jordan method to Solve*

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 2 \\ x_1 + 2x_2 - x_3 &= 1 \\ -x_1 + x_2 - 2x_3 &= -2 \end{aligned}$$

Solution $\overline{A} = \begin{pmatrix} 1 & -1 & 3 & |2 \\ 1 & 2 & -1 & |1 \\ -1 & 1 & -2 & |-2 \end{pmatrix} R_2 - R_1, R_3 + R_1 \rightarrow \begin{pmatrix} 1 & -1 & 3 & |2 \\ 0 & 3 & -4 & |-1 \\ 0 & 0 & 1 & |0 \end{pmatrix}$

$\frac{1}{3}R_2 \rightarrow \begin{pmatrix} 1 & -1 & 3 & |2 \\ 0 & 1 & \frac{-4}{3} & |\frac{-1}{3} \\ 0 & 0 & 1 & |0 \end{pmatrix} R_2 + \frac{4}{3}R_3, R_1 - 3R_3 \rightarrow$

$\begin{pmatrix} 1 & -1 & 0 & |2 \\ 0 & 1 & 0 & |\frac{-1}{3} \\ 0 & 0 & 1 & |0 \end{pmatrix} R_1 + R_2 \rightarrow \begin{pmatrix} 1 & 0 & 0 & |\frac{5}{3} \\ 0 & 1 & 0 & |\frac{-1}{3} \\ 0 & 0 & -1 & |0 \end{pmatrix}$

Solution $\begin{pmatrix} \frac{5}{3} \\ \frac{-1}{3} \\ 0 \end{pmatrix}$.

Remark 1.3.7. If in the process of solving a linear system by Gauss elimination or Gauss-Jordan elimination, and the left hand of a row is reduced to a zero row but the right hand is nonzero then the system is inconsistent. That is if we get a row of the form $\begin{bmatrix} 0 & 0 & ... & 0|1 \end{bmatrix}$, then the system is inconsistent.

Definition 1.3.8. *The variable that correspond to the leading one are called the leading variables, and the remaining variables, if any, are called free variables.*

Remark 1.3.9. A linear system with a free variable is either inconsistent or has infinite number of solutions.

Example 1.3.6. $\begin{matrix} x_1 - x_2 + x_3 = 2 \\ x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + x_2 = 3 \end{matrix}$ *is consistent with x_3 free, so it has infinite solutions*

but, $\begin{matrix} x_1 - x_2 + x_3 = 2 \\ x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + x_2 = 1 \end{matrix}$ *is inconsistent with x_3 free. Why?*

Overdetermined and underdetermined systems

Definition 1.3.10. . *An $m \times n$ linear system is called **underdetermined** system if $m < n$, and it is called **overdetermined** if $m > n$*

Remark 1.3.11. An underdetermined linear system always has a free variable, so it is either inconsistent or it has infinite solutions.

Remark 1.3.12. An overdetermined linear system can't tell. (all cases possible).

Definition 1.3.13. . *An $m \times n$ linear system is called **homogeneous** if all right hand of every equation is zero. That is the augmented matrix $\overline{A}$ of the linear system is of the form $\overline{A} = (A|0)$, that is $(b_1 = b_2 = ... = b_k = 0)$.*

Remark 1.3.14.

1. A homogeneous linear system is always consistent with $x_1 = x_2 = ... = x_n = 0$ is a solution called the zero solution or the trivial solution

2. A homogeneous linear system is either has a unique solution (the zero solution) if it has no free variables or it has infinite solutions if it has a free variable.

3. An underdetermined homogeneous linear system always has infinite solutions.

1.4 1.3+1.4 Matrix Algebra.

Recall that a matrix is any array of objects.

A row or a column of a matrix is called a vector and the $i-$row of a matrix A is denoted by $\overrightarrow{a_i}$ and the $i-$column of a matrix A is denoted by a_i. The set of all row matrices or the set of all column matrices is called the Euclidean space denoted by either R^n or $R^{1\times n}$

An $m\times n$ matrix is usually represented by its columns as $a = (a_1, a_2, ..., a_n)$ or by its rows as $A = \begin{pmatrix} \overrightarrow{a_1} \\ \overrightarrow{a_2} \\ . \\ . \\ . \\ \overrightarrow{a_m} \end{pmatrix}$

Definition 1.4.1. ***Equality of matrices**: Two matrices A, B are equal iff they the same size and the corresponding entries are equal*

Operations on matrices .
1. Scalar multiplication.

Definition 1.4.2. *Let A be an $m \times n$ matrix, $c \in R$. Then $cA = B$, where $b_{ij} = ca_{ij}, \forall i, j$*

2. Matrix addition.

Definition 1.4.3. *Let A, B be $m \times n$ matrices. Then $A + B = C$, where $c_{ij} = a_{ij} + b_{ij}, \forall i, j$*

Properties of addition and scalar multiplication

Theorem 1.4.4. *Let A, B be an $m \times n$ matrices, $\alpha, \beta \in R$. Then*

1. $\alpha(A + B) = \alpha A + \alpha B$
2. $\alpha\beta(A) = \alpha(\beta A)$
3. $A + B = B + A$
4. A+(B+C)=(A+B)+C
5. $A + 0 = 0 + A = A$

6. $A + -A = -A + A = 0$

2. Matrix multiplication.

Definition 1.4.5. *Let A be $m \times n$, B an $n \times k$ matrices. Then $AB = C$, where $c_{ij} = \sum_{k=1}^{k=n} a_{ik}b_{kj}$*

Example 1.4.1. $\begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -3 & 5 \\ 3 & 2 & 0 \end{pmatrix}$

Properties of matrix multiplication

Theorem 1.4.6. *Let A be $m \times n$, B an $n \times k$, C be $k \times l$ matrices, $\alpha, \beta \in R$. Then*

1. $\alpha(AB) = A(\alpha B)$
2. $AB \neq BA$
3. $A(BC) = (AB)C$
4. *Let A be $m \times n$, B, C an $n \times k$. Then $A(B + C) = AB + AC$*

Remark 1.4.7. If A an $m \times n$, B an $n \times k$, then $AB = (Ab_1, Ab_2, ..., Ab_k)$ using the columns of B or

$$AB = \begin{pmatrix} \overrightarrow{a_1}B \\ \overrightarrow{a_2}B \\ . \\ . \\ . \\ \overrightarrow{a_n}B \end{pmatrix} \text{ using the rows of } A$$

Remark 1.4.8. 1. If $AB = AC$ then we cannot conclude $B = C$

Example 1.4.2. *Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. Then $AB = AC$ but $B \neq C$*

2. If $AB = 0$ then we cannot conclude $A = 0$ or $B = 0$

Example 1.4.3. *Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ but neither A nor B is a zero matrix*

Remark 1.4.9.

1. If A, B an $n \times n$ upper triangular matrices, then AB is an upper triangular matrix
2. If A, B an $n \times n$ lower triangular matrices, then AB is a lower triangular matrix
3. If A, B an $n \times n$ diagonal matrices, then AB is a diagonal triangular matrix

Linear systems and matrices

A linear system with augmented matrix $\overline{A} = (A|b)$ can be written in matrix multiplication as $Ax = b$, where $A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ . & & & \\ . & & & \\ . & & & \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{pmatrix}$

Also, we can write the linear system as $Ax = b$ as $a_1x_1 + a_2x_2 + ... a_nx_n = b$

Remark 1.4.10.

1. A vector X_0 is a solution of a linear system $Ax = b$ iff $AX_0 = b$
2. If vectors X_0, X_1 are solutions of a linear system $Ax = b$. Then $\alpha X_0 + \beta X_1$ is a solution iff $\alpha + \beta = 1$. Since $A(\alpha X_0 + \beta X_1) = b$ iff $\alpha b + \beta b = b$ iff $\alpha + \beta = 1$.
3. If vectors X_0, X_1 are solutions of a homogeneous linear system $Ax = 0$. Then $\alpha X_0 + \beta X_1$ is a solution of $Ax = 0$ for any $\alpha, \beta \in R$. Since $A(\alpha X_0 + \beta X_1) = \alpha AX_0 + \beta AX_1 = 0$.

Definition. Linear combinations

Definition 1.4.11. *Let $a_1, a_2, ..., a_k \in R^n, c_1, c_2, ..., c_k \in R$. Then a vector $c_1a_1 + c_2a_2 + ... + c_ka_k$ is called a linear combination of the vectors $a_1, a_2, ..., a_k$*

Example 1.4.4. 1. $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ *is a linear combination of* $a = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, *and* $b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, *since* $v = 1a + 0b$

2. *Is* $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ *a linear combination of* $a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, *and* $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

   Solution. *Let* $v = c_1 a + c_2 b$, *if the system has a solution then* v *is a linear combination of* a, b

   So let $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

   We get the linear system whose augmented matrix $\begin{pmatrix} 1 & 1 & | & 1 \\ 0 & 1 & | & 2 \end{pmatrix}$ *and this system has a unique solution* $c_2 = 2, c_1 = -1$, *so* v *is a linear combination of* a *and* b.

Theorem. Consistency of the linear system.

Theorem 1.4.12. *A linear system* $Ax = b$ *is consistent iff* b *is a linear combination of the columns of* A.

Proof. $\Rightarrow$ Suppose the system $Ax = b$ is consistent, so there exist real numbers $c_1, c_2, ..., c_n$ such that $A \begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_n \end{pmatrix} = b$. So $c_1 a_1 + c_2 a_2 + ... + c_n a_n = b$, and so b is a linear combination of the columns of A

$\Leftarrow$ Suppose b is a linear combination of the columns of A, so there exist real numbers $c_1, c_2, ..., c_n$ such that $c_1 a_1 + c_2 a_2 + ... + c_n a_n = b$. But the last equation is $c_1 a_1 + c_2 a_2 + ... + c_n a_n = A \begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_n \end{pmatrix} = b$. So $\begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_n \end{pmatrix}$ is a solution of $Ax = b$ □

Remark 1.4.13. The proof of the consistency of the linear systems shows that if b is a linear combination of the columns of the A, then the coefficients of the column of A is a solution of the linear system $Ax = b$.

Example 1.4.5. 1. *Let $A_{3\times3}, Ax = b$, and $b = 2a_1 - 3a_2 + a_3$, then the system is consistent and* $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ *is a solution of $Ax = b$, but we don't know if the system has a unique solution or infinite solutions*

2. *Let $A_{2\times3}, Ax = b$, and $b = 2a_1 - 3a_2 + a_3$, then the system is consistent and* $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ *is a solution of $Ax = b$, but the system is undetermined and consistent, so it has infinite solutions.*

3. *Let $A3 \times 3, Ax = 0$, and $0 = 2a_1 + 5a_3$, then the system is consistent and* $\begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$ *is a non zero solution of the homogeneous $Ax = 0$, so it has infinite solutions.*

Remark 1.4.14. If b can be written in more than way as a linear combination of the columns of the A, then the linear system $Ax = b$ has infinite solutions.

Transpose.

Definition 1.4.15. *The transpose of an $m \times n$ matrix A is an $n \times m$ matrix B such that $b_{ij} = b_{ji}, \forall i, j$ denoted by A^t*

Definition 1.4.16. *An $n \times n$ matrix A is symmetric iff $A^t = A$, and it is skew-symmetric iff $A^t = -A$*

Properties of the transpose

1. $(A^t)^t = A$
2. $(A + B)^t = A^t + B^t$
3. $(cA)^t = cA^t, \forall c \in R$
4. $(AB)^t = B^t A^t$

5. If an $n \times n$ matrices A, B are symmetric, then $A + B$ is symmetric.

6. If an $n \times n$ matrices A is symmetric, then $cA, \forall c \in R$ is symmetric.

Example 1.4.6. *If* $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$*, then* $A^t = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Special matrices
A matrix A is called

1. A zero matrix iff all entries are zeros $(a_{ij=0}, \forall i, j$

2. An upper triangular iff A is a square matrix such that $a_{ij} = 0, \forall i > j$

3. A lower triangular iff A is a square matrix such that $a_{ij} = 0, \forall i < j$

4. A diagonal iff A is a square matrix such that $a_{ij} = 0, \forall i \neq j$

5. Identity matrix denoted by I_n is a diagonal matrix such that $\delta_{ii} = 1, \delta_{ij} = 0, \forall i \neq j$

Nonsingular(invertible) matrices.

Definition 1.4.17. *A square $n \times n$ matrix A is said to be nonsingular or invertible iff there exists a square $n \times n$ matrix B such that $AB = BA = I_n$, and B is called the inverse of A denoted by A^{-1}, that is $AA^{-1} = A^{-1}A = I_n$. A none invertible matrix is called singular.*

Properties of the inverse

1. Inverse if it exists is unique

2. $(A^{-1})^{-1} = A$

3. $(AB)^{-1} = B^{-1}A^{-1}$

4. If A is invertible, then A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$

Remark 1.4.18. 1. The sum of invertible need not be invertible.

2. If A is invertible and $AB = AC$ then $B = C$

3. From number 3, if A, B are $n \times n$ invertible matrices then AB is invertible

1.5 Elementary matrices and inverses

Definition 1.5.1. *A matrix E is called an elementary matrix if it is obtained from I_n by only one row operation.*

Example 1.5.1. *1.* $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2. $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3. $C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Types of (ROW) elementary Matrices:

1. **Type I:** E is obtained from I_n by interchanging any two rows of I_n : C

2. **Type II:** E is obtained from I_n by multiplying any rows of I_n by a nonzero constant: B

3. **Type III:** E is obtained from I_n by adding a multiple of one row of I_n to another row of I_n : A

Remark 1.5.2. Similarly, we have column elementary matrices by performing similar operations on the columns of the identity matrix. But we focus on the row elementary matrices

Theorem 1.5.3. *Multiplying a matrix A from left by an elementary matrix is the same as performing a row operation on A of the same type*

Theorem 1.5.4. *Multiplying a matrix A from right by a column elementary matrix is the same as performing a column operation on A of the same type*

Definition 1.5.5. *A matrix A is called row equivalent to a matrix B if A is obtained from B by performing a sequence of row operations on A. Equivalently, if $A = E_1E_2...E_kB$, where $E_i's$ are elementary matrices.*

Theorem 1.5.6. *Any elementary matrix E is invertible and E^{-1} is an elementary matrix of the same type by reversing the operation on I_n*

Theorem 1.5.7. *Equivalent conditions for nonsingularity of a matrix A. Let A be a square $n \times n$ matrix. Then the following are equivalent (FAE)*

1. *Ais nonsingular*

2. *$Ax = 0$ has only the zero solution(trivial solution)*

3. *A is row equivalent to I_n*

Proof. $1 \Rightarrow 2$. Let A be nonsingular and $Ax = 0$. Multiply both sides by A^{-1} from left, we get $A^{-1}Ax = A^{-1}0$. So $Ix = 0$, so $x = 0$ is the only solution of $Ax = 0$.

$2 \Rightarrow 3$. Suppose A is not row equivalent to IA_n, so the reduced row echelon form of A has a free variable and so $Ax = 0$ has infinite solutions.

$3 \Rightarrow 1$. Let A be row equivalent to I_n, so there exist elementary matrices $E_1, E_2, ..., E_k$ such that $E_1, E_2, ..., E_kA = I$. So $E_1E_2...E_k = A^{-1}$, and so A is invertible. □

Remark 1.5.8. If A is nonsingular, then $Ax = b$ has a unique solution which is $x = A^{-1}b$

Remark 1.5.9. The above theorem gives a strategy to find the inverse of a square matrix if it exist, since if A is nonsingular then A is row equivalent to I_n. So there exist elementary matrices $E_1, ..., E_k$ such that $E_k...E_2E_1A = I_n$, and so $E_k...E_1I_n = A^{-1}$. That is if we perform row operations on A to change it into I_n, then performing the same row operations on the identity matrix I_n we get A^{-1}

$(A|I) \rightarrow$ row operations$(I|A^{-1})$

Example 1.5.2. *Find the inverse of* $A = \begin{pmatrix} 1 & -1 & 3 \\ 1 & 2 & -1 \\ -1 & 4 & -2 \end{pmatrix}$

Solution. $\left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ -1 & 4 & -2 & 0 & 0 & 1 \end{array}\right) R_2 - R_1, R_3 + R_1 \rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 3 & -4 & -1 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 1 \end{array}\right)$

$R_3 - R_2 \rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 3 & -4 & -1 & 1 & 0 \\ 0 & 0 & 5 & 2 & -1 & 1 \end{array}\right) \frac{1}{5}R_3 \rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 3 & -4 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{5} & \frac{-1}{5} & \frac{1}{5} \end{array}\right)$

$R_2+4R_3, R_1-3R_3 \rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & \frac{-1}{5} & \frac{3}{5} & \frac{-3}{5} \\ 0 & 3 & 0 & \frac{3}{5} & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 1 & \frac{2}{5} & \frac{-1}{5} & \frac{1}{5} \end{array}\right)$

$\frac{1}{3}R_2 \rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & \frac{-1}{5} & \frac{3}{5} & \frac{-3}{5} \\ 0 & 1 & 0 & \frac{1}{5} & \frac{1}{15} & \frac{4}{15} \\ 0 & 0 & 1 & \frac{2}{5} & \frac{-1}{5} & \frac{1}{5} \end{array}\right)$

$R_1+R_2 \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{2}{5} & \frac{-1}{3} \\ 0 & 1 & 0 & \frac{1}{5} & \frac{1}{15} & \frac{4}{15} \\ 0 & 0 & 1 & \frac{2}{5} & \frac{-1}{5} & \frac{1}{5} \end{array}\right)$

So A is nonsingular and $A^{-1} = \begin{pmatrix} 0 & \frac{2}{5} & \frac{-1}{3} \\ \frac{1}{5} & \frac{1}{15} & \frac{4}{15} \\ \frac{2}{5} & \frac{-1}{5} & \frac{1}{5} \end{pmatrix}$

Definition 1.5.10. *A matrix A is called row equivalent to a matrix B if A is obtained from B by performing a sequence of row operations on A. Equivalently, if $A = E_1E_2...E_kB$, where $E_i's$ are elementary matrices.*

Remark 1.5.11. If in the processes of performing row operations on $(A|I)$, one rows of A is reduced to a zero row then A is singular

Example 1.5.3. $\begin{pmatrix} 1 & -1 & 3 \\ 1 & 2 & -1 \\ -2 & 2 & -6 \end{pmatrix}$ *has no inverse*

A rule only for 2×2 matrices.

Let A be 2×2 matrix,say, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then A is invertible iff $ad-cb \neq 0$, and $A^{-1} = \frac{1}{ad-cb}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Prove this

Example 1.5.4. *1.* $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ *is invertible and* $A^{-1} = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}$,

2. $B = \begin{pmatrix} 2 & 12 \\ 1 & 6 \end{pmatrix}$ *is not invertible.*

Triangular Factorization.

Definition 1.5.12. *If a matrix A is reduced into an upper triangular matrix using* ***row operations of type III only*** *then A has a triangular factorization $A = LU$, where U is upper triangular and L is unit lower triangular.*

Remark 1.5.13. Not every matrix has an LU factorization

Example 1.5.5. 1. *Find the LU factorization of* $A = \begin{pmatrix} 1 & -1 & 3 \\ 1 & 2 & -1 \\ -1 & 1 & -2 \end{pmatrix}$ *if it exists*

Solution. $\begin{pmatrix} 1 & -1 & 3 \\ 1 & 2 & -1 \\ -1 & 1 & -2 \end{pmatrix} R_2 - R_1, R_3 + R_1 \rightarrow \begin{pmatrix} 1 & -1 & 3 \\ 0 & 3 & -4 \\ 0 & 0 & 1 \end{pmatrix} = U$,

So, $E_1 = I_3(R_2 - R_1) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$E_2 = I_3(R_3 + R_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

So, $E_2E_1A = U$. *So* $A = (E_2E_1)^{-1}U$.

That is $L = (E_2E_1)^{-1}I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$. *That is to get L we perform row operations on* I_3 *opposite to the row operations on* A*(*L *is a lower triangular matrix with* $1's$ *in the main diagonal and if we perform the row operation* $R_i - \alpha R_j$*, then* $l_{ij} = \alpha$*), so* $L = I_3(R_2 + R_1, R_3 - R_1) \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

2. *Find the LU factorization of* $A = \begin{pmatrix} 0 & -1 & 3 \\ 1 & 2 & -1 \\ -1 & 1 & -2 \end{pmatrix}$ *if it exists*

Solution. *A has no LU factorization since we can't use the first row to terminate the first entries in the lower rows.*

Exercises.

1. Answer the following by true false

   (a) IfA, B are $n \times n$ matrices and $AB = 0$, then $(A + B)^2 = A^2 + B^2$

   (b) If A has an LU-factorization and A is singular then U is singular.

   (c) Let A, B be $n \times n$ symmetric matrices. If $AB = BA$ then AB is symmetric.

   (d) If A is symmetric and skew symmetric then A must be a zero matrix.(A is skew symmetric if $A^T = -A$).

   (e) If the system $Ax = b$ is consistent then b is a linear combinations of the columns of A.

   (f) If A, B are square $n \times n$ matrices and $AB = 0$, then A or B is singular.

   (g) If A, B are square $n \times n$ matrices and AB is singular then A or B is singular.

   (h) If the coefficient matrix of the system $AX = b$ is singular then the system has infinitely many solutions.

   (i) In the linear system $Ax = 0$, if $0 = a_1$ then the system has a unique solution.

   (j) If the row echelon form of the matrix A involves a free variable, then the linear system $Ax = b$ has infinitely many solutions.

   (k) A square matrix A is nonsingular iff its RREF is the identity matrix.

   (l) If $AB = AC$, $A \neq 0$, then $C = B$.

   (m) In the linear system $AX = b$, if b is the first column of A then the system has infinitely many solutions.

2. Solve the linear system $Ax = b$ whose augmented matrix $\begin{pmatrix} 1 & 2 & 3 & | & 1 \\ -1 & 1 & 3 & | & 0 \\ 1 & 1 & 0 & | & -1 \end{pmatrix}$ using both Gauss Elimination Method and Gauss Jordan Elimination method

3. Solve the linear system $\begin{array}{c} x_1 - x_2 + x_3 = 2 \\ x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + x_2 = 3 \end{array}$

4. Solve the linear system $\begin{array}{c} x_1 - x_2 + x_3 = 2 \\ x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + x_2 = 1 \end{array}$

5. Let $\overline{\mathbf{A}} = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ -1 & 1 & b & a \\ 1 & 1 & 0 & 1 \end{array}\right)$. Find the conditions on a, b so the system is (1) consistent, (2) inconsistent

6. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix}$. Find the inverse of A if it exists.

7. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix}$. Find LU-factorization of A

8. Let $Ax = b$ be linear system where A is an $n \times n$ singular. What can you say a bout $Ax = 0$, and $Ax = b, b \in R^n$.

9. Let $Ax = 0$ be linear homogeneous system where A is an 2×3 nonzero matrix. If $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ are two solutions of the homogeneous system. Find all solutions of $Ax = 0$.

10. Let $Ax = b$ be linear system where A is an 2×3 nonzero matrix. If $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ are two solutions of the linear system $Ax = b$. Find all solutions of $Ax = b$, and a nonzero solutions of $Ax = 0$.

11. If A, B are square $n \times n$ nonzero matrices such that $AB = 0$. Show that A and B are singular.

12. If A, B are nonzero square $n \times n$ matrices such that $AB = 0$. Show that the homogeneous system $Ax = 0$ must have infinite solutions.

13. If A, B are nonzero matrices such that $AB = 0$. Show that the homogeneous system $Ax = 0$ must have infinite solutions.

14. If A, B are $n \times n$ symmetric. Then AB is symmetric iff $AB = BA$.

15. If the reduced row echelon form of the augmented matrix of the linear system $Ax = b$ is $\begin{pmatrix} 1 & 0 & 1 & |2 \\ 0 & 1 & 2 & |-1 \\ 0 & 0 & 0 & |0 \end{pmatrix}$, and $a_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $a_2 = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$.

    Find b.

Chapter 2

Determinants

2.1 Determinants

Definition 2.1.1. *If A is an $n \times n$ matrix, then the determinant of A is denoted by $det(A)$ or $|A|$. If A is a 1×1, say, $A = (a_{11})$. Then $det(A) = a_{11}$, and if A is a 2×2 matrix, say, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then $det(A) = a_{11}a_{22} - a_{21}a_{12}$*

Example 2.1.1.

1. *Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$, then $det(A) = 2(4) - 3(-1) = 11$*

2. *Let $A = \begin{pmatrix} -1 \end{pmatrix}$, then $|A| = -1$*

2.1.1 Cofactor Method

Definition 2.1.2. *Let A be an $n \times n$ matrix, and let M_{ij} be an $(n-1) \times (n-1)$ matrix obtained from A by deleting the $i - th$ row and the $j - th$ column of A. Then the minor of a_{ij} is the determinant of M_{ij}, and the cofactor of a_{ij} denoted by $A_{ij} = (-1)^{(i+j)}|M_{ij}|$.*

Definition 2.1.3. *Let A be an $n \times n$ matrix. Then we define the determinant of A by $|A| = \left\{ \begin{matrix} a_{11}, & n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + ... + a_{1n}A_{1n}, & n \geq 2 \end{matrix} \right\}$. (This is called the expansion of the determinant of A along the first row of A).*

Theorem 2.1.4. *Let A be an $n \times n$ matrix. Then $|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + ... + a_{in}A_{in}$. (This is called the expansion of the determinant of A along the $i-th$ row of A).*

Theorem 2.1.5. *Let A be an $n \times n$ matrix. Then $|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + ... + a_{nj}A_{nj}$. (This is called the expansion of the determinant of A along the $j-th$ column of A).*

Mathematical induction

If $S(n)$ is a mathematical statement then this statement is true for every n iff

1. $S(1)$ is true
2. Assume $S(k)$ is true
3. Prove $S(k+1)$ is true

Theorem 2.1.6. *Let A be an $n \times n$ matrix. Then $|A| = |A^t|$.*

Proof. 1. If $n = 1$, then $A^t = A$. So $|A| = |A^t|$

2. Assume the result is true for $n = k$. (That is, if A is of size $k \times k$, then $|A| = |A^t|$)

3. Let A be of size $(k+1) \times (k+1)$ and expand $|A|$ on the first row. So $|A| = a_{11}A_{11} + ... + a_{1n}A_{1n} = a_{11}A^t_{11} + ... + a_{1,k+1}A^t_{1,k+1} = a^t_{11}A^t_{11} + ... + a^t_{k+1,1}A^t_{k+1,1} = |A^t|$ □

Theorem 2.1.7. *Let A be an $n \times n$ matrix.*

1. *If A has a zero row or a zero column, then $|A| = 0$*
2. *If A has two identical rows or two identical columns, then $|A| = 0$*

Proof. 1. If A has a zero row, say $i-th$ row, then compute $|A|$ using that row, so $|A| = a_{i1}A_{i1} + ... + a_{in}A_{in} = 0 + 0 + ... + 0 = 0$

2. We use mathematical induction on the size of the matrix A 1. If $n = 2$, say $A = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$. Then $|A| = ab - ab = 0$

   2. Assume the result is true for $n = k$ (that is if A is of size $k \times k$ with two identical rows then $|A| = 0$

3. Let A be of size $(k+1) \times (k+1)$ with two identical rows then expand $|A|$ on any row distinct from the identical rows, say row j. So $|A| = a_{j1}A_{j1} + ... + a_{j,k+1}A_{j,k+1}$, but now each $A_{j,l}$ is a determinant of matrix of size k with identical rows i, j, so $A_{j,l} = 0, i = 1, ..., k+1$

□

Theorem 2.1.8. *If A is a triangular matrix, then $|A| = a_{11}a_{22}...a_{nn}$. (That is the determinant is the product of the entries in the main diagonal)*

Proof. We use mathematical induction on the size of the matrix A 1. If $n = 2$, say $A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$. Then $|A| = a_{11}a_{22}$

2. Assume the result is true for $n = k$.

3. Let A be of size $(k+1) \times (k+1)$ upper triangular then expand $|A|$ on the first column. So $|A| = a_{11}A_{11}$, but now each A_{11} is a determinant of an upper triangular matrix of size k. So $|A| = a_{11}a_{22}...a_{nn}$

□

2.2 Properties of the determinant

Theorem 2.2.1. *Let A be an $n \times n$ matrix. Then*
$a_{i1}A_{j1} + a_{i2}A_{j2} + ... + a_{in}A_{jn} = \left\{ \begin{array}{ll} 0, & i \neq j \\ |A| & i = j \end{array} \right\}.$

Proof. If $i = j$, then $a_{i1}A_{j1} + a_{i2}A_{j2} + ... + a_{in}A_{jn} = a_{i1}A_{i1} + a_{i2}A_{i2} + ... + a_{in}A_{in} = |A|$. If $i \neq j$, let $A*$ be the matrix obtained from A by replacing the $j-$th row of A by its $i-$th row. Expand the determinant of A^* using the $j-$th row. Since A^* has two identical rows, so $|A^*| = 0$. So, $0 = |A^*| = a^*_{j1}A*_{j1} + a^*_{j2}A^*_{j2} + ... + a^*_{jn}A^*_{jn} = a_{i1}A_{j1} + a_{i2}A_{j2} + ... + a_{in}A_{jn}$ □

2.2.1 Row operations

Theorem 2.2.2. *Let A be a quare matrix and B is obtained form A by* ***only one row*** *operation. Then*

1. ***Type I:****Type I (B is obtained by interchanging two rows of A. Then $|B| = -|A|$*

2. ***Type II:*** *B is obtained form A by multiplying one row only of A by a nonzero constant, say, α. Then $|B| = \alpha|A|$*

3. ***Type III:*** *B is obtained by adding a multiple of one row of A to another row of A. Then $|B| = |A|$*

Proof. By MI on the size of the matrix (Exercise)

The above theorem is equivalent to the next theorem. □

Theorem 2.2.3. *Let A be a square matrix and $B = EA$, E is an elementary matrix. Then*

1. ***Type I:*** *If E is an elementary matrix of type I (E is obtained by interchanging two rows of I_n. Then $|B| = -|A|$*

2. ***Type II:*** *If E is an elementary matrix of type II (E is obtained by multiplying one row only of I_n by a nonzero constant, say, α. Then $|B| = \alpha|A|$*

3. ***Type III:*** *If E is an elementary matrix of type III (E is obtained by adding a multiple of one row to another row of I_n. Then $|B| = |A|$*

A special case of the above theorem, we get the next two theorems

Theorem 2.2.4. *Let E be an elementary matrix. Then*

1. ***Type I:****If E is an elementary matrix of type I (E is obtained by interchanging two rows of I_n. Then $|E| = -1$*
2. ***Type II:*** *If E is an elementary matrix of type II (E is obtained by multiplying one row only of I_n by a nonzero constant, say, α. Then $|E| = \alpha$*
3. ***Type III:*** *If E is an elementary matrix of type III (E is obtained by adding a multiple of one row to another row of I_n. Then $|E| = 1$*

Theorem 2.2.5. *Let E be an elementary matrix, and A be a matrix of the same size of E. Then $|EA| = |E||A|$*

Theorem 2.2.6. *Let $E_1, ..., E_k$ be elementary matrices. Then $|E_1...E_k| = |E_1|...|E_k|$*

Proof. $|E_1...E_k| = |E_1|...|E_k| = |E_1||E_2...E_k| = |E_1||E_2||E_3...E_k| = |E_1|...|E_k|$.
OR we can use MI. □

Theorem 2.2.7. *A square matrix A is nonsingular iff $|A| \neq 0$*

Proof. Let A be nonsingular. So A is row equivalent to I_n. Thus there exist elementary matrices $E_1, ..., E_k$ such that $A = E_1, ..., E_k I_n$. So $|A| = |E_1||E_2|...|E_k| \neq 0$
Conversely, if $|A| \neq 0$. Then we use row operations to change A into RREF. So there exist elementary matrices $E_1, ..., E_k$ and a matrix U in RREF such that $A = E_1, ..., E_k U$. Since $|A| \neq 0$. So $|U| \neq 0$, since all $E_i's$ are invertible, and $|A| = |E_1|...|E_k||U|$. So $U = I_n$, and so A is invertible. □

Theorem 2.2.8. *If A, B are $n \times n$ matrices, then $|AB| = |A||B|$*

Proof. If A is singular then $|A| = 0$, and so AB is singular, and so $|AB| = |A||B| = 0$.
So let A be nonsingular and so A is row equivalent to I_n. Thus $|AB| = |E_1 E_2 ... E_k B| = |E_1||E_2|...|E_k||B| = |E_1 E_2 ... E_k||B| = |A||B|$. □

2.3 Adjoint and Cramer's rule

Definition 2.3.1. *Let A be $n \times n$ matrix. The adjoint of A denoted by $adj(A)$ is an $n \times n$ whose $ij-th$ entry is A_{ji} that is $adj(A) = C^t$, where* $C = \left\{ \begin{array}{cccc} A_{11} & A_{12} & ... & A_{1n} \\ A_{21} & A_{22} & & A_{2n} \\ A_{n1} & A_{n2} & & A_{nn} \end{array} \right\}$

Example 2.3.1. *Find $adj(A)$ of $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$*

Solution $A_{11} = 4, A_{12} = -3, A_{21} = 1, A_{22} = 2$, *so* $adj(A) = \begin{pmatrix} 4 & 1 \\ -3 & 2 \end{pmatrix}$

Theorem 2.3.2. *Let A be $n \times n$ matrix. Then adjoint of $Aadj(A) = |A|I_n$*

Proof. The $ij-$th entry of $Aadj(A) = a_{i1}A_{j1} + ... + a_{in}A_{jn} = \left\{ \begin{array}{cc} |A|, & i = j \\ 0, & i \neq j \end{array} \right\} = |A|I_n$ □

Theorem 2.3.3. *Let A be $n \times n$ nonsingular matrix. Then $A^{-1} = \frac{adj(A)}{|A|}$*

Proof. Since A is nonsingular, so A^{-1} exists. Multiply $Aadj(A) = |A|I_n$ from left by A^{-1}, so $adj(A) = |A|A^{-1}$. Since $|A| \neq 0$, so $A^{-1} = \frac{adj(A)}{|A|}$ □

The above theorem gives another way to find the inverse if it exists, **called the cofactor method, or the adjoint method**.

Cramer's Rule

Theorem 2.3.4. *Let A be $n \times n$ nonsingular matrix. Then the solutions of $Ax = b$ are given by adjoint of $x_i = \frac{|A_{ib}|}{|A|}$, where A_{ib} is a matrix obtained from A by replacing the $i-th$ column of A by the column b*

Proof. Since A is nonsingular, so A^{-1} exists. Multiply $Ax = b$ from right by A^{-1}, so $x = A^{-1}b = \frac{adj(A)}{|A|}b$. So $x = \frac{1}{|A|} \begin{pmatrix} A_{11} & A_{21} & ... & A_{n1} \\ A_{12} & A_{22} & ... & A_{n2} \\ ... & ... & ... & ... \\ A_{1i} & A_{2i} & ... & A_{ni} \\ ... & ... & ... & ... \\ A_{1n} & A_{2n} & ... & A_{nn} \end{pmatrix} b$. So

$x_i = \frac{1}{A}(b_1A_{1i} + b_2A_{i2} + ... + b_nA_{ni} = \frac{|A_{ib}|}{|A|}$ □

Remark 2.3.5. Cramer's rule is not practical since it can be used only if the system has a unique solution, also the number of operation are very large since it involves computing the determinants.

Example 2.3.2. *Use Cramer's rule to solve the linear systems*

1. $Ax = b$, *where*
   $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$
2. $Ax = b$, *where*
   $A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Solution (1) $|A| = 11$, so A is nonsingular, so $A_{1b} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}, A_{2b} = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$ So $|A_{1b}| = 11, |A_{2b}| = 0$. Thus $x_1 = \frac{|A_{1b}|}{|A|} = 1, x_2 = \frac{|A_{2b}|}{|A|} = 0$

(2)$|A| = 0$, so A is singular, so we can't use Cramer's rule

Exercises.

1. Answer the following by true false

   (a) If A, B are $n \times n$ matrices. Then A, B are nonsingular iff AB is nonsingular

   (b) If E is an elementary matrix. Then $E^{-1} = E$

   (c) If E is an elementary matrix. Then $|E^{-1}| = |E|$

   (d) Let A, B be $n \times n$ equivalent matrices. Then $|A| = |B|$.

   (e) Let A be $n \times n$. Then $|\alpha A| = \alpha |A|$.

   (f) Let A be $n \times n$. Then $|adj(A)| = |A|$.

2. Use Cramer's method to solve the linear system $Ax = b$ whose augmented matrix $\begin{pmatrix} 1 & 2 & 3 & | & 1 \\ -1 & 1 & 3 & | & 0 \\ 1 & 1 & 0 & | & -1 \end{pmatrix}$

3. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix}$. Find the inverse of A using the adjoint.

4. If A is a square $n \times n$ matrix. Show that A is nonsingular iff $adj(A)$ is nonsingular.

5. If A is a square $n \times n$ matrix. Show that $|adj(A)| = |A|^{n-1}$.

6. If A is a square $n \times n$ matrix. Show that $adj(adj(A)) = |A|^{n-2}A$.

7. Let $\mathbf{adj}(\mathbf{A}) = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix}$. Find A.

Sample First Exam

Q1 :(20 points) Answer the following statements by true or false

(a) If A, B are square $n \times n$ nonzero matrices such that $AB = 0$, then A and B are singular.

(b) If $A = LU$ is the LU-factorizaton and A is singular then U is singular.

(c) If A, B, AB are $n \times n$ symmetric matrices then $AB = BA$.

(d) If A is symmetric and skew symmetric then A must be a zero matrix.(A is skew symmetric if $A^T = -A$).

(e) If A is an $n \times n$ nonsingular matrix then $det(adj(A)) = (det(A))^{n-1}$.

(f) If the system $Ax = b$ is consistent then b is a linear combinations of the columns of A.

(g) If A, B are square $n \times n$ matrices and AB is singular then A and B are singular.

(h) If A is row equivalent to B then $\det(A) = \det(B)$.

(i) If the coefficient matrix of the system $AX = b$ is singular then the system has infinitely many solutions.

(j) In the linear system $Ax = b$, if b is a linear combinations of the columns of A then the system has a unique solution.

(k) If the row echelon form of the matrix A involves a free variable, then the linear system AX=b has infinitely many solutions.

(l) a square matrix A is nonsingular iff its row echelon form is the identity matrix.

(m) If $AB = AC$, $A \neq 0$, then $A = B$.

(n) In the linear system $Ax = b$, if b is the first column of A, then the system has infinitely many solutions.

(o) If $det(A) = det(B)$, then $A = B$

Q2 (20points) Let $\mathbf{A} = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ -1 & 1 & 3 & 4 \\ 1 & 2 & \alpha & \beta \end{array} \right)$ be the Augmented matrix of a linear system. Find the values of α, β so that the system

(i) is consistent

(ii) inconsistent

Q3 (20points) Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix}$ be the coefficient matrix of a linear system $AX = b$. Find

(i) LU-factorization of A

(ii) Use LU-factorization to solve the system $AX = b$, where $b = (1, 1, 1)^t$

Q4 (20points) Let A be an $n \times n$ nonsingular matrix

(i) Show that $adj(adj(A)) = |A|^{n-2}A$.

(ii) Let A, B be $n \times n$ square symmetric matrices. Show that $AB = BA$ iff AB is symmetric

Chapter 3

Vector Spaces

3.1 Definition and Examples

Definition 3.1.1. *A none empty set V with two operations $+ : V \times V \to V$, $\cdot : R \times V \to V$ is called a vector space iff the following holds*

1. $a + b \in V$, $\forall a, b \in V$
2. $\alpha a \in V$, $\forall \alpha \in R, \forall a \in V$
3. $0 \in V$ *such that* $0 + a = a + 0 = a, \forall a \in V$
4. $\forall a \in V$, $-a \in V$, *and* $a + -a = -a + a = 0$
5. $\forall a, b, c \in V$, $a + (b + c) = (a + b) + c$
6. $\forall a, b \in V$, $a + b = b + a$
7. $(\alpha\beta)a = \alpha(\beta a), \forall \alpha, \beta \in R, \forall a \in V$
8. $\alpha(a + b) = \alpha a + \alpha b, \forall \alpha \in R, \forall a, b \in V$
9. $(\alpha + \beta)a = \alpha a + \beta a, \forall \alpha, \beta \in R, \forall a \in V$
10. $1 \cdot a = a$, $\forall a \in V$

Example 3.1.1.

1. *R with usual addition and multiplication is a vector space*

2. $M_{n\times m} \equiv R^{n\times m}$ *is the set of all* $m \times n$ *matrices under addition and scalar multiplication of matrices is a vector space*

3. *The set of all real valued functions under addition and scalar multiplication of functions:* $(f+g)(x) = f(x)+g(x), (\alpha f)(x) = \alpha f(x)$ *is a vector space*

   The zero polynomial denoted by $Z(x)$ ***or*** $0(x)$ ***is of degree zero***

4. $C[a,b] = \{f : [a,b] \to R : f$ *is continuous on* $[a,b]\}$ *under addition and scalar multiplication of functions:* $(f+g)(x) = f(x)+g(x), (\alpha f)(x) = \alpha f(x)$ *is a vector space*

5. $C^n[a,b] = \{f : [a,b] \to R : f^{(n)}$ *is continuous on* $[a,b]\}$ *under addition and scalar multiplication of functions:* $(f+g)(x) = f(x)+g(x), (\alpha f)(x) = \alpha f(x)$ *is a vector space*

6. $P_n = \{f(x) = a_{n-1}x^{n-1}+...a_1x+a_0, a_i's \in R\}$ *under addition and scalar multiplication of functions:* $(f+g)(x) = f(x)+g(x), (\alpha f)(x) = \alpha f(x)$ *is a vector space*

7. $V = \{f(x) : deg(f) = 3\}$ *under addition and scalar multiplication of functions:* $(f+g)(x) = f(x)+g(x), (\alpha f)(x) = \alpha f(x)$ *is not a vector space*

8. Q, Z *are not vector spaces.*

9. $\{(0,a) : a \in R\}$ *under addition and scalar multiplication of matrices is a vector space*

10. $\{(1,a) : a \in R\}$ *under addition and scalar multiplication of matrices is not a vector space*

Theorem 3.1.2. *Let* V *be a vector space. Then*

1. $0v = \boldsymbol{0}, \forall v \in V$

2. *If* $x+y = \boldsymbol{0}$*, then* $y = -x$

3. $-1 \cdot v = -v$

Proof. 1. If $0 = 0+0$, so $(0+0)v = 0v$. Thus $0v+0v = 0v$, add to both sides $-0v$. So, $0v+0v+-0v = 0v = +-0v$. Thus $0v+\mathbf{0} = \mathbf{0}$. Hence, $0v = \mathbf{0}$

2. Add $-x$ to both sides of $x + y = \mathbf{0}$. So, $-x + x + y = -x + \mathbf{0}$. Thus $y = -x$.

3. $0 = 1 + -1$, so $(1 + -1)v = 0v = \mathbf{0}$ by 1. Thus $1v + -1v = \mathbf{0}$, so $v + -1v = \mathbf{0}$. Add to both sides $-v$. So, $-v + v + -1v == -v$. Thus $-1v = -v$.

$\square$

3.2 Subspaces and spanning sets

Definition 3.2.1. *A none empty subset S of a vector space V is called a subspace of V iff the following holds*

1. $x + y \in S, \forall x, y \in S$
2. $\alpha \cdot x \in S, \forall x \in S, \forall \alpha \in R$

Theorem 3.2.2. *Let S be a subspace of a vector space V. Then $\mathbf{0} \in S$*

Proof. Since S is a subspace of V, so $S \neq \phi$. Let $x \in S$. So $0x = \mathbf{0} \in S$ □

Remark 3.2.3. Let S be a subset of a vector space V. If $\mathbf{0} \notin S$, then S is not a subspace of V

Example 3.2.1.

1. $S = \{A_{n\times n} : |A| = 0\}$, $V = \{A_{n\times n}\}$, *is not subspace, since the sum of two singular need not be singular*
2. $S = \{A_{n\times n} : |A| \neq 0\}$, $V = \{A_{n\times n}\}$, *is not subspace, since the sum of two nonsingular need not be nonsingular. Also the zero matrix is not nonsingular*
3. $S = \{A_{m\times n} : a_{11} = 0$, $V = \{A_{m\times n}\}$ *is a subspace*
4. $S = \{A_{n\times n} : A^t = A\}$, $V = \{A_{n\times n}\}$ *is a subspace*
5. $S = \{A_{n\times n} : A$, *is triangular* $\}$ *is not a subspace*
6. $S = C^3[a, b]$, $V = C[a, b]\}$ *is a subspace*
7. $S = P_n$, $V = C(R)\}$ *is a subspace, where, $C(R)$ is the set of all continuous functions on R.*
8. $S = \{(a, b)^t : a + b = 1, a, b \in R\}$, $V = \{(a, b)^t : a, b \in R\}$ *is not a subspace*
9. $\{(1, a)^t : a \in R\}$, $V = \{(a, b)^t : a, b \in R\}$ *is not a subspace*
10. $\{(0, a)^t : a \in R\}$, $V = \{(a, b)^t : a, b \in R\}$ *is a subspace*

Proof. HW □

The Null Space of a Matrix

Definition 3.2.4. *Let A be $m \times n$ matrix. The null space of A denoted by $N(A) = \{x \in R^n : Ax = 0\}$*

Theorem 3.2.5. *Let A be $m \times n$ matrix. Then $N(A)$ is a subspace of R^n*

Proof.

1. $N(A) \neq \phi$ since $\mathbf{0} \in N(A)$
2. Let $x, y \in N(A)$. Then $Ax = 0, Ay = 0$, so $A(x+y) = Ax + Ay = 0 + 0 = 0$. So $x + y \in N(A)$
3. Let $x \in N(A), \alpha \in R$. Then $Ax = 0$, so $A(\alpha x) = \alpha Ax = \alpha(0) = 0$, so $\alpha x \in N(A)$ So $N(A)$ is a subspace of R^n

□

Linear Combinations.

Definition 3.2.6. *Let V be a vector space and let $v_1, v_2, ..., v_k \in V, c_1, c_2, ..., c_k \in R$. Then a vector $c_1v_1 + c_2v_2 + ... + c_kv_k$ is called a linear combination of the vectors $v_1, v_2, ..., v_k$. The set of all linear combinations of $v_1, v_2, ..., v_k$ is called the span of $v_1, v_2, ..., v_k$ which is denoted by $Span(v_1, v_2, ..., v_k)$*

Example 3.2.2.

1. *$v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a linear combination of $a = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and $b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, since $v = 1a + 0b$*
2. *Is $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ a linear combination of $a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$*

   Solution. *Let $v = c_1a + c_2b$, if the system has a solution then v is a linear combination of a, b*

   So let $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

   We get the linear system whose augmented matrix $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \end{array}\right)$ and this system has a unique solution $c_2 = 2, c_1 = -1$, so v is a linear combination of a and b.

Theorem 3.2.7. *Let V be a vector space and let $v_1, v_2, ..., v_k \in V$. Then $Span(v_1, v_2, ..., v_k)$ is a subspace of V*

Proof. 1. $Span(v_1, v_2, ..., v_k) \neq \phi$, since $\mathbf{0} = 0v_1 + 0v_2 + ... + 0v_k \in Span(v_1, v_2, ..., v_k)$

2. Let $x, y \in Span(v_1, v_2, ..., v_k)$. Then $x = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_k v_k$, $y = c_1 v_1 + c_2 v_2 + ... + c_k v_k$. So $x + y = (\alpha_1 + c_1)v_1 + (\alpha_2 + c_2)v_2 + ... + (\alpha_k + c_k)v_k \in Span(v_1, v_2, ..., v_k)$

3. $x = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_k v_k, \alpha \in R$. Then $\alpha x = \alpha\alpha_1 v_1 + \alpha\alpha_2 v_2 + ... + \alpha\alpha_k v_k \in Span(v_1, v_2, ..., v_k)$ So $Span(v_1, v_2, ..., v_k)$ is a subspace of V

□

Theorem 3.2.8. *Let V be a vector space and let S, T be subspaces of V. Then*

1. *$S \bigcap T$ is a subspace of V*
2. *$S \bigcup T$ is not always a subspace of V*
3. *$S + T = \{x + y : x \in S, y \in T\}$ is a subspace of V*

Proof. HW □

Definition 3.2.9. *Let V be a vector space. A set of $v_1, v_2, ..., v_k \in V$ is called a spanning set of V iff every vector $v \in V$ is linear combination of $v_1, v_2, ..., v_k$. That is $V = Span(v_1, v_2, ..., v_k)$*

Notation

Let $V = R^n$, and let e_i be an $n \times 1$ column matrix with 1 in the ith component and zero otherwise, that is e_i is the $i - th$ column of I_n.

Example 3.2.3. 1. *$e_1, e_2, ..., e_n$ span R^n, this is called the standard spanning set for R^n. Since if $x = (x_1, x_2, ..., x_n)^t \in R^n$, then $x = x_1 e_1 + ... + x_n e_n$*

2. *$1, x, ..., x^{n-1}$ span P_n, this is called the standard spanning set for P_n. Since if $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_{n-1} x^{n-1} \in P_n$, then $f(x) = a_0(1) + (a_1)x + (a_2)x^2 + ... + (a_{n-1})x^{n-1}$*

Example 3.2.4. *1. Does* $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ *span* R^3

2. Is $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *a spanning set for* R^2

3. Is $v_1 = x, v_2 = 1, v_3 = 2x - 1$ *a spanning set for* P_2

Solution.

1. Let $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, and let $v = c_1v_1 + c_2v_2 + c_3v_3$. This system is not always consistent, why

2. Let $v = \begin{pmatrix} a \\ b \end{pmatrix}$, and let $v = c_1v_1 + c_2v_2 + c_3v_3$. This system is always consistent, why

3. Let $v = ax + b \in P_2$, and let $v = c_1v_1 + c_2v_2 + c_3v_3$. This system is always consistent, why

Linear System Revisited

Theorem 3.2.10. *Let* A *be an* $m \times n$ *matrix, and let the linear system* $Ax = b$ *be consistent with* x_0 *a solution. Then* y *is a solution of* $Ax = b$ *iff* $y = x_0 + z$*, where* $z \in N(A)$

Proof. $A(x_0 + z) = Ax_0 + Az = b + 0 = b$, so $x_0 + z$ is a solution of $Ax = b$. Also, if y is another solution of $Ax = b$, then $y - x_0$ is a solution of $Ax = 0$, since $A(y - x_0) = Ay - Ax_0 = b - b = 0$. So $y - x_0 \in N(A)$. That is there exists $z \in N(A)$ such that $y - x_0 = z$. So $y = x_0 + z$

□

3.3 Linear Independence

Definition 3.3.1. *Let V be a vector space. A set of $v_1, v_2, ..., v_k \in V$ is called linearly independent (li) iff the only solution of $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_k v_k = 0$ is the zero solution $\alpha_1 = \alpha_2 = ... = \alpha_k = 0$. Otherwise, they are linearly dependent (ld).*

Example 3.3.1. *1. Is $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ li*

2. Is $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ li

3. Is $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ li

4. Is $v_1 = x, v_2 = 1, v_3 = 2x - 1$ li

Solution.

1. Let $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$. So we solve the homogeneous system $v_1 = \begin{pmatrix} 1 & 1 & 0 & | & 0 \\ 2 & 2 & 0 & | & 0 \\ 3 & 2 & 1 & | & 0 \end{pmatrix}$, and the coefficient matrix is singular so it has infinite solutions. So the vectors are ld

2. Let $c_1 v_1 + c_2 v_2 = 0$. This system has aunique solution. So the vectors are li

3. Let $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$. This system is underdetermined so it has infinite solutions. So the vectors are ld

4. Let $v = c_1 v_1 + c_2 v_2 + c_3 v_3$. This system is underdetermined so it has infite solutions. So the vectors are ld

Remark 3.3.2. Any set of vectors that contain the zero vectro are ld. Why?

Theorem 3.3.3. *A set of vectors $v_1, v_2, ..., v_k$ in a vector space V are ld iff one of them is a linear combination of the remaining set of vectors.*

Proof. $\Rightarrow$. Say v_1 is a linear combination of $v_2, ..., v_k$. So there exist constants $c_2, ..., c_k \in R$ such that $v_1 = c_2v_2 + ... + c_kv_k$, and so $(-1, c_2, ..., c_k)$ is a nonzero solution of $\alpha_1 v_1 + ... + \alpha_k v_k = 0$. So $v_1, v_2, ..., v_k$ are ld.

$\Leftarrow$. Let $v_1, v_2, ..., v_k$ be linearly dependent, so $\alpha_1 v_1 + ... + \alpha_k v_k = 0$ has a nonzero solution,say, $(c_1, c_2, ..., c_k)$, and at least one of the $c_i's$ is nonzero,say, c_1. So $v_1 = \frac{-c_2}{c_1}v_2 + ... + \frac{-c_k}{c_1}v_k$, and so v_1 is a linear combination of $v_2, ..., v_k$ □

Theorem 3.3.4. *A set of vectors $v_1, v_2, ..., v_k$ in a vector space V are li iff every vector $v \in Span(v_1, v_2, ..., v_k)$ is uniquely written as a linear combination of $v_1, v_2, ..., v_k$.*

Proof. $\Rightarrow$. Suppose $v \in Span(v_1, v_2, ..., v_k)$ are not uniquely written as a linear combination of $v_2, ..., v_k$,say, $v = \alpha_1 v_1 + ... + \alpha_k v_k = \beta_1 v_1 + ... + \beta_k v_k$. So $(\alpha_1 - \beta_1)v_1 + ... + (\alpha_k - \beta_k)v_k = 0$. So $(\alpha_1 - \beta_1, ..., \alpha_k - \beta_k)$ is a nonzero solution of $c_1v_1 + ... + c_kv_k = 0$. So $v_1, v_2, ..., v_k$ are ld.

$\Leftarrow$. Let $\alpha_1 v_1 + ... + \alpha_k v_k = 0$. Since, $0v_1 + ... + 0v_k = 0$, and $0 \in Span(v_1, v_2, ..., v_k)$, so $\alpha_1 = 0, .., \alpha_k = 0$. Thus, $v_1, ..., v_k$ are li. □

Theorem 3.3.5. *A set of vectors $v_1, v_2, ..., v_n$ in a vector space R^n are li iff the matrix $A = (v_1, v_2, ..., v_n)$ is nonsingular.*

Proof. $v_1, v_2, ..., v_n$ are li iff $\alpha_1 v_1 + ... + \alpha_n v_n = 0$ has only the zero solution iff A is nonsingular. □

Theorem 3.3.6. *Let a set of vectors $f_1, f_2, ..., f_n$ in $C^{n-1}[a, b]$ be ld, then* $A = \begin{pmatrix} f_1 & ... & f_n \\ f_1' & ... & f_n' \\ ... & ... & \\ ... & ... & \\ f_1^{n-1} & ... & f_n^{n-1} \end{pmatrix}$ *is singular.*

Proof. Suppose $f_1, f_2, ..., f_n$ in $C^{n-1}[a, b]$ be ld, then there exist constants $c_1v_1 + ... + c_kv_n$ not all zeros such that $c_1f_1 + ... + c_nf_n = 0$. Take all $(n-1)$ derivatives of the previous equation, we get $c_1v_1 + ... + c_kv_n$ is a nonzero solution of $\begin{pmatrix} f_1 & ... & f_n \\ f_1' & ... & f_n' \\ ... & ... & \\ ... & ... & \\ f_1^{n-1} & ... & f_n^{n-1} \end{pmatrix} X = 0$. So $A = \begin{pmatrix} f_1 & ... & f_n \\ f_1' & ... & f_n' \\ ... & ... & \\ ... & ... & \\ f_1^{n-1} & ... & f_n^{n-1} \end{pmatrix}$ is singular. □

Definition 3.3.7. *Let $f_1, f_2, ..., f_n$ in $C^{n-1}[a,b]$. The Wronskian of $f_1, f_2, ..., f_n$ denoted by $W(f_1, f_2, ..., f_n)$ is defined by $W(f_1, f_2, ..., f_n) = |A|$, where* $A = \begin{pmatrix} f_1 & ... & f_n \\ f_1' & ... & f_n' \\ ... & ... & \\ ... & ... & \\ f_1^{n-1} & ... & f_n^{n-1} \end{pmatrix}$.

Theorem 3.3.8. *Let $f_1, f_2, ..., f_n$ in $C^{n-1}[a,b]$. If $W(f_1, f_2, ..., f_n) \neq 0$ for some $x \in [a,b]$. Then $f_1, f_2, ..., f_n$ are li*

Proof. Follows from the above theorem by contrapositive. □

Remark 3.3.9. Let $f_1, f_2, ..., f_n$ in $C^{n-1}[a,b]$. If $W(f_1, f_2, ..., f_n) = 0$. Then test fails

Example 3.3.2.

1. *$x^2, 2x^2$ are ld, but $W(x^2, 2x^2) = 0$*
2. *$x^2, x|x|$ over $C^1[-1,1]$ are li, but $W(x^2, x|x|) = 0$*
3. *Is $x, x^2, x-1, sin^2x, cos^2x, e^x$ li?*

3.4 Basis and Dimension

Definition 3.4.1. *A set of vectors $v_1, v_2, ..., v_n$ in a vector space V is a basis for V iff:*

1. $v_1, v_2, ..., v_n$ *span* V
2. $v_1, v_2, ..., v_n$ *are li*

Example 3.4.1.

1. $e_1, e_2, ..., e_n$ *is a basis for* R^n *called the standard basis*
2. $1, x, ..., x^{n-1}$ *is a basis for* P_n *called the standard basis*
3. *Is* E_{ij} *such that* $e_{ij} = 1$ *and* 0, *otherwise is a standard basis for* $R^{m\times n}$
4. $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ *is a basis for* R^3
5. $1 + x, x + 3$ *is a basis for* P_2

Theorem 3.4.2. *Let a set of vectors $v_1, v_2, ..., v_n$ be a spanning set for V. If $w_1, w_2, ..., w_m \in V, m > n$. Then $w_1, w_2, ..., w_m$ are ld*

Proof. Let $\alpha_1 w_1 + ... + \alpha_m w_k = 0$. Since $v_1, v_2, ..., v_n$ span V, so for each w_i, there exist $c_{ij}, j = 1, ..., n \in R$ such that $w_i = c_{i1}v_1 + c_{i2}v_2 + ... + c_{in}v_n$. Now substitute $w_i = c_{i1}v_1 + c_{i2}v_2 + ... + c_{in}v_n$ in $\alpha_1 w_1 + ... + \alpha_m w_k = 0$, we get an $n \times m$ homogeneous system with $m > n$. So the system has a nonzero solution. So, $w_1, w_2, ..., w_m$ are ld. □

Theorem 3.4.3. *Let V be a vector space with two basis $v_1, v_2, ..., v_n$, and $w_1, w_2, ..., w_m$. Then $m = n$.*

Proof. Since $v_1, v_2, ..., v_n$ span V, and $w_1, w_2, ..., w_m$ are li, so by previous theorem $m \le n$. Similarly, since $w_1, w_2, ..., w_m$ span V, and $v_1, v_2, ..., v_n$ are li, so by previous theorem $n \le m$. So $m = n$. □

Definition 3.4.4. *Let V be a nonzero vector space. If V has a finite basis $v_1, v_2, ..., v_n$, then V is called finite dimensional vector space with dimension n, written $dim(V) = n$. The zero vector space $\{\mathbf{0}\}$ has dimension zero with basis ϕ. Otherwise, V is called infinite dimensional, written $dim(V) = \infty$.*

Example 3.4.2.

1. R^n *has dimension* n

2. *P_n has dimension n*

3. *$R^{m\times n}$ has dimension $n.m$*

4. *$C^n[a,b]$ has dimension ∞*

Theorem 3.4.5. *Let V be a vector space with dimension n. Then the following are equivalent(FAE)*

1. *$v_1, v_2, ..., v_n$ is a basis*

2. *$v_1, v_2, ..., v_n$ span*

3. *$v_1, v_2, ..., v_n$ are li.*

Proof. $1 \Rightarrow 2$. Clearly, if $v_1, v_2, ..., v_n$ is a basis for V, then $v_1, v_2, ..., v_n$ span V.

$2 \Rightarrow 3$. So let $v_1, v_2, ..., v_n$ span V but ld. So one of them is a linear combination of the others, say, so $v_1 \in Span(v_2, ..., v_n)$. If $v_2, ..., v_n$ are li, then $v_2, ..., v_n$ is a basis for V with $n-1$ vectors, a contradiction. So $v_2, ..., v_n$ are ld, and so one of this set is a linear combination of the remaining set, say, $v_2 \in Span(v_3, ..., v_n)$. Similarly, if $v_3, ..., v_n$ are li, then $v_3, ..., v_n$ is a basis for V with $n-2$ vectors, a contradiction. We continue the same process and at some stage, we must get a set which is li and span V that is it is a basis with fewer than n vectors, a contradiction. So, $v_1, v_2, ..., v_n$ are li. $2 \Rightarrow 3$ Let $v_1, v_2, ..., v_n$ be li, but does not span V, so there exists a nonzero vector $u \in V$ and $u \notin span(v_1, v_2, ..., v_n)$. So, $u, v_1, v_2, ..., v_n$ are ld, and so there exist $c_i, j = 1, ..., n+1 \in R$ not all zeros such that $c_1v_1+c_2v_2+...+c_nv_n+c_{n+1}u = 0$. But $c_{n+1} \neq 0$ for if $c_{n+1} = 0$, then $v_1, v_2, ..., v_n$ are ld, a contradiction. But, $c_{n+1} \neq 0$ implies u is a linear combination of $v_1, v_2, ..., v_n$, a contradiction. So, $v_1, v_2, ..., v_n$ span V. □

Remark 3.4.6. Let V be a vector space with dimension $n > 0$. Then

1. A set of $v_1, v_2, ..., v_m, m > n$ is ld.

2. A set of $v_1, v_2, ..., v_m, m < n$ can not span V.

3. A li set of $v_1, v_2, ..., v_m, m < n$ can be extended to a basis for V.

4. A spanning set of $v_1, v_2, ..., v_m, m > n$ can be reduced(pared down) to a basis for V.

Remark 3.4.7.

1. $[e_1, e_2, ..., e_n]$ is called the standard basis for R^n.
2. $[1, x, ..., x^{n-1}]$ is called the standard basis for P_n.

3.5 Change of Basis

Definition 3.5.1. *Let V be a vector space with an ordered basis $B = [v_1, v_2, ..., v_n]$, and $v \in V$. Then there exist $c_1..., c_n \in R$ such that $v = c_1v_1 + ... + c_nv_n$. The vector $(c_1, ..., c_n)^t \in R^n$ is called the coordinate vector of v with respect to the basis B denoted by $[v]_B$*

Now, let V be a vector space with two basis $B = [v_1, v_2, ..., v_n]$, and $S = [w_1, w_2, ..., w_n]$, and $v \in V$. Is there a relation between $[v]_B$, and $[v]_S$. We start with a simple example.

Example 3.5.1. *Let $V = R^2$ with basis $[u_1, , u_2]$, say, $u_1 = (u_{11}, u_{12})^t, u_2 = (u_{21}, u_{22})^t$ and let $v = (x_1, x_2)^t \in R^2$, then there exist $c_1, c_2 \in R$ such that $v = (x_1, x_2)^t = c_1u_1 + c_2u_2$. So, $\begin{pmatrix} u_1 & u_2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. That is $\begin{pmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Let $U = [u_1, u_2]$, then U is called the transition matrix from the basis B into the standard basis $[e_1, e_2]$, and U^{-1} is the transition matrix from the standard basis $[e_1, e_2]$ into the basis $U = [u_1, u_2]$*

In general, if $B = [u_1, u_2], S = [w_1, w_2]$ are any two none standard basis of R^2, let $U_1 = (u_1, u_2)$ be the transition matrix from $B = [u_1, u_2]$ into the standard basis $[e_1, e_2]$, and $U_2 = (w_1, w_2)$ be the transition matrix from $S = [w_1, w_2]$ into the standard basis $[e_1, e_2]$. Then the transition matrix from B into S is $U = U_2^{-1}U_1$

Theorem 3.5.2. *Let V be a finite dimensional vector space with dimension n. If $B = [v_1, v_2, ..., v_n]$, $S = [w_1, w_2, ..., w_n]$. Then the transition matrix from the basis B into the basis S is the $n \times n$ nonsingular matrix*
$U = ([v_1]_S, [v_2]_S, ..., [v_n)_S]$.

Remark 3.5.3. Let V be a finite dimensional vector space with basis B, and let $v_1, .., v_k \in V$. Then

1. $[v_1 + v_2 + ... + v_k]_B = [v_1]_B + [v_2]_B + ... + [v_k]_B$
2. $v_1, .., v_k$ are li iff $[v_1]_B, [v_2]_B, ..., [v_k]_B$ are li.

3.6 Row space, Column space, Rank, and Nullity

Definition 3.6.1. *Let A be $m \times n$ matrix. Then*

1. *The row space of A is the subspace of R^n spanned by the rows of A denoted by $R(A)$, that is $R(A) = span[\overrightarrow{a_1}, \overrightarrow{a_2}, ..., \overrightarrow{a_m})$*

2. *The column space of A is the subspace of R^m spanned by the columns of A denoted by $C(A)$, that is $C(A) = span[a_1, a_2, ..., a_m)$*

3. *The null space of A is the subspace of R^n which is the solution of the homogeneous system $Ax = 0$ denoted by $N(A)$.*

4. *The nullity of A denoted by $Null(A) = dim(N(A)$.*

5. *The rank of A denoted by $rank(A) = dim(C(A)$.*

Theorem 3.6.2. *Let A, B be $m \times n$ equivalent matrices. Then $R(A) = R(B)$. Consequently, if U is the REF of A, then $R(A) = R(U)$*

Proof. If A, B are row equivalent, then the rows of A are linear combinations of the rows of B, so $R(A) \subset R(B)$. Similarly, the rows of B are linear combinations of the rows of A, so $R(B) \subset R(A)$. So, $R(A) = R(B)$. □

Theorem 3.6.3. *Let A be $m \times n$ matrix. Then $dim(R(A)) = dim(C(A))$.*

Theorem 3.6.4. *Let A be $m \times n$ matrix. Then $Rank(A) + Null(A) = n$.*

Proof. Let U be the REF of A. Then $dim(R(A)) = dim(R(U))$ is the number of leading variables, and the $dim(N(A)) = dim(N(U))$ is the number of free variables. So, $Rank(A) + Null(A) = n$ □

Theorem 3.6.5. *Let A be $m \times n$ matrix. If U is the REF of A, then the columns of A that correspond to the leading $1's$ in U is a basis for $C(A)$*

Example 3.6.1. *Find $R(A), C(A), N(A), null(A), rank(A)$ of* $A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 0 \\ 1 & 2 & 1 & -1 \end{pmatrix}$

Solution. REF of A is $U = \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. So,

1. Basis for $R(A)$ is $(1, 2, 0, -1), (0, 0, 1, 0), (0, 0, 0, 2)$
2. Basis for $C(A)$ is $(1, 1, 2, 1)^t, (0, 0, 0, 1)^t, (-1, 1, 0, -1)^t$
3. $Rank(A) = 3$
4. $Null(A) = 1$
5. $N(A) = (-2\alpha, \alpha, 0, 0)^t, \alpha \in R$ with basis $(-2, 1, 0, 0)^t$

Back to the linear system $Ax = b$

Recall that the consistency theorem of the linear system $Ax = b$ says that the linear system $Ax = b$ is consistent iff b is a linear combination of the columns of A iff $b \in C(A)$. Consequently we get the following theorem

Theorem 3.6.6. *Let A be $m \times n$ matrix, $b \in R^m$. Then*

1. *The linear system $ax = b$ is consistent for every $b \in R^m$ iff $C(A) = R^m$.*
2. *The columns of A are li, iff the linear system $Ax = b$ is either inconsistent or has a unique solution.*

Theorem 3.6.7. *Let A be $n \times n$ matrix. Then A is nonsingular iff the columns of A form a basis for R^n.*

Sample Exam Q1: (45 points)

(1) Let V be a vector space. Mark each of the following statements by true or false.

(a) For any $v \in V$, $-v \in V$. T

(b) For any $v, w \in V$, $v \cdot w \in V$. F

(c) For any $v \in V$, $2v \in V$. T

(d) For any $v \in V$, $0v \in R$. F

(e) V could be equal ϕ. F

(2) Let V be a vector space, v_1, v_2, v_3, v_4 span V. Mark each of the following statements by true or false.

(a) $dim(V) = 4$. F

(b) $dim(V) \geq 4$. F

(c) $dim(V) \leq 4$. T

(d) any set of more than 5 vector in V are linearly dependent. T

(e) Any basis of V has exactly 4 vectors. F

(3) Let V be a vector space, $dim(V) = 5$. Mark each of the following statements by true or false.

(a) If v_1, v_2, v_3, v_4, v_5 in V, then v_1, v_2, v_3, v_4, v_5 is a basis for V. F

(b) If v_1, v_2 in V, then v_1, v_2 are linearly independent. F

(c) If v_1, v_2 in V, then v_1, v_2 can't span V. T

(d) If v_1, v_2, v_3, v_4, v_5 in V, and $v \in V$, then $v \in Span(v_1, v_2, v_3, v_4, v_5)$. F

(e) If B is a basis for V, then $v_B \in R^5$. T

(4) Let A be 3×3 matrix such that $|A| = 0$. Mark each of the following statements by true or false.

(a) Rank(A) $= 3$. F

(b) Rank(A) < 3. T

(c) $\text{Null}(A) = 1$. F
(d) $N(A) = \{\mathbf{0}\}$. F
(e) $\text{Null}(A) = 0$. F

(5) Let A, B be $n \times n$ nonzero matrices such that $AB = 0$. Mark each of the following statements by true or false.

(a) $Ax = 0$ has a nonzero solution. T
(b) $Ran(A) = Rank(B)$. F
(c) $Ran(B) \leq Null(A)$. T
(d) $Ran(A) \leq Null(B)$
(e) $Ax = 0$ has only the zero solution. F

(6) Let A, B be subspaces of a vector space V. Mark each of the following statements by true or false.

(a) $A \bigcap B$ is a subspace of V. T
(b) $A \bigcup B$ is a subspace of V. F
(c) $A + B = \{x + y; x \in A, y \in B\}$ is a subspace of V. T
(d) $2A = \{2x : x \in A\}$ is a subspace of V. T
(e) $A \bigcap B \neq \phi$. T

Q2: (10 points) Let A, B be subspaces of a vector space V.

(a) Show that $A \bigcap B$ is a subspace of V
See notes
(b) Let A be $m \times n$ matrix. Show that $N(A)$ is a subspace of R^n.
See notes

Q3: (10 points)

Let $V = P_3$ and let $S = \{f \in V : f(0) = 0, f(1) = 0\}$.

(a) Show S is a subspace of V
Do it
(b) Find a basis for S $x^2 - x$

Q4: (20 points)

Let $A = \begin{pmatrix} 1 & 1 & 2 & 1 & 4 \\ 1 & -1 & 2 & -1 & 6 \\ 3 & 1 & 6 & 1 & 14 \end{pmatrix}$. Find

(a) A basis for row space of A

(b) A basis for column space of A

(c) A basis for null space of A

(d) $Rank(A)$

Q5: (25 points) Let $T : R^2 \to R^3$ defined by $T(x, y) = (x - z, y - x, x - y)$

(a) Show T is a linear transformation

(b) Find the matrix representation of T with respect to the standard basis of R^2, R^3

(c) Find a basis for $ImmT$

(d) Find a basis for $kerT$

(e) Find all $v \in R^2 : T(v) = (1, 1, 1)$

Q6: (10 points) Let $V = P_2$, $B = [1 - x, 2 + x], F = [1 + 2x, 2 - 3x]$

(a) Find the transition matrix S from B into F

(b) Use the transition matrix S to find the vector v where $v_{[B]} = (2, 5)^t$

Chapter 4

Linear Transformations

4.1 Definitions, Examples, and Basic Properties

Definition 4.1.1. *Let V, W be vector spaces. A mapping (a function) $L : V \to W$ is called a linear transformation (LT) iff*

(a) $L(v_1 + v_2) = L(v_1) + L(v_2), \forall v_1, v_2 \in V$

(b) $L(\alpha v) = \alpha L(v), \forall v \in V, \forall \alpha \in R$

Example 4.1.1. (a) $L : R^2 \to R^2, L((x, y)^t) = (x, -y)^t$ *is a linear transformation (reflection on X-axis)*

(b) $L : R^2 \to R^2, L((x, y)^t) = (-x, y)^t$ *is a linear transformation (reflection on y-axis)*

(c) $L : R^2 \to R^2, L((x, y)^t) = (x^2, y)^t$ *is not a linear transformation*

(d) *Let V be a vector space, and let $v_0 \in V$, $L : V \to V, L(v) = v + v_0$ is a linear transformation iff $v_0 = \mathbf{0}$*

(e) $L : R^2 \to R^2, L((x, y)^t) = (x, y + 1)^t$ *is not a linear transformation*

Theorem 4.1.2. *Let V, W be vector spaces, and let $L : V \to W$ be a linear transformation. Then*

(a) $L(0_V) = 0_W$

(b) $L(v_1 - v_2) = L(v_1) - L(v_2), \forall v_1, v_2 \in V$

(c) $L(\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n) = \alpha_1 L(v_1) + \alpha_2 L(v_2) + ... + \alpha_n L(v_n), \forall v_1, v_2, ..., v_n \in V, \forall \alpha_1, \alpha_2, ..., \alpha_n \in R$

Proof. (a) $L(0_V) = L(00_V) = 0L(0_V) = 0_W$

(b) $L(v_1 - v_2) = L(v_1 + -1v_2) = L(v_1) + L(-1v_2) = L(v_1) - 1L(v_2) = L(v_1) - L(v_2), \forall v_1, v_2 \in V$

(c) BY MI

□

Remark 4.1.3. Let V, W be vector spaces, and let $L : V \to W$ be a mapping. If $L(0_V) \neq 0_W$, then L is not a linear transformation

Theorem 4.1.4. *Let V, W be vector spaces, and let $L : V \to W$ be a mapping. Then L is a linear transformation iff $L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2), \forall v_1, v_2 \in V, \forall \alpha, \beta \in R$*

Example 4.1.2. *(a) $L : C[0, 1] \to R^2, L(f(x)) = \begin{pmatrix} \int_0^1 f(x)dx \\ f(0) \end{pmatrix}$ is a linear transformation*

(b) $L : P_3 \to P_2, L(f(x)) = f'(x)$ is a linear transformation

(c) Let A be $m \times n$ matrix, and let $L : R^n \to R^m, L(X) = AX, \forall X \in R^n$ is a linear transformation

Remark 4.1.5. Let V, W be vector spaces, and let V be a finite dimensional vector space with basis $B = [v_1, ..., v_n]$, and let $L : V \to W$ be a LT. Then L is completely determined by the basis B. That if $L(v_1), ..., L(v_n)$ are given then for any $v \in V, v = c_1 v_1 + ... + c_n v_n$, and so $L(v) = c_1 L(v_1) + ... + c_n L(v_n)$

4.1.1 Kernel and images

Definition 4.1.6. *Let $L : V \to W$ be a linear transformation. Then*

(a) The kernel of L denoted by $kerL = \{v \in V : L(v) = 0_W\}$

(b) The image (or the range)of L denoted by $ImmL$ (or $L(V)$ or R_L) is defined by $L(V) = \{w \in W : w = L(v)$ for some $v \in V\}$

Theorem 4.1.7. *Let $L : V \to W$ be a linear transformation. Then*

(a) $kerL$ is a subspace of V

(b) $L(V)$ is a subspace of W

Proof. (a) 1. $0 \in kerL$, since $L(0) = 0$

2. Let $v_1, v_2 \in KerL$. Then $L(v_1) = L(v_2) = 0$, so $L(v_1 + v_2) = L(v_1) + L(v_2) = 0 + 0 = 0$. Hence, $v_1 + v_2 \in KerL$

3. Let $v \in kerL, \alpha \in R$. Then $L(\alpha v) = \alpha L(v) = \alpha 0 = 0$, so $\alpha v \in KerL$. Thus $KerL$ is a subspace of V

(b) 1. $0 \in ImmL$, since $L(0) = 0$

2. Let $w_1, w_2 \in ImmL$. So there exist $v_1, v_2 \in V$ such that $w_1 = L(v_1), w_2 = L(v_2)$, so $w_1 + w_2 = L(v_1) + L(v_2) = L(v_1 + v_2) \in L(V)$.

3. Let $w \in ImmL, \alpha \in R$. So there exist $v \in V$ such that $w = L(v)$. Then $\alpha w = \alpha L(v) = L(\alpha v) \in ImmL$. Thus $ImmL$ is a subspace of W

□

Example 4.1.3. *Find the kernel and image of the following linear transformations*

(a) $L : P_2 \to R^2, L(f(x)) = \begin{pmatrix} \int_0^1 f(x)dx \\ f(0) \end{pmatrix}$

(b) $L : P_3 \to P_2, L(f(x)) = f'(x)$

(c) $L : R^4 \to R^2, L((x_1, x_2, x_3, x_4)^t) = (x_1 + x_2 + x_3, x_4)^t$

Solution:

(a) 1. $KerL = \{f(x) = ax + b : L(f(x)) = \begin{pmatrix} \int_0^1 f(x)dx \\ f(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

So $\begin{pmatrix} \frac{a}{2} + b \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. So $a = b = 0$. Thus $KerL = Z(x) = 0(x)$

2. $ImmL = \{(x, y)^t \in R^2 : (x, y)^t = L(ax + b) = \begin{pmatrix} \frac{a}{2} + b \\ b \end{pmatrix} = a\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. So, a basis for $L(P_2)$ is $\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ which is a basis for R^2. So, $ImmL = R^2$

(b) $L : P_3 \to P_2, L(f(x)) = f'(x)$

1. $KerL = \{f(x) : L(ax^2 + bx + c) = 2ax + b = 0\}$. So $a = b = 0$. Thus $KerL = f(x) = c$

2. $ImmL = \{(g(x) \in P_2 : g(x) = L(ax^2 + bx + c) = 2ax + b$. So, a basis for $L(P_3)$ is $2x, 1$ which is a basis for P_2. So, $ImmL = P_2$

(c) 1. $KerL = \{(x_1, x_2, x_3, x_4)^t : L((x_1, x_2, x_3, x_4)^t) = (x_1 + x_2 + x_3, x_4)^t = (0, 0)^t\}$. So, $x_4 = 0, x_1 = -x_2 - x_3$. So $KerL = (-x_2 - x_3, x_2, x_3, 0)^t$, and so a basis for $KerL$ is $(-1, 1, 0, 0)^t, (-1, 0, 1, 0)^t$

2. $ImmL = (x, y)^t = (x_1 - x_2 + x_3, x_4)^t = x_1(1, 0)^t + x_2(-1, 0)^t + x_3(1, 0)^t + x_4(0, 1)^t$ with basis $(1, 0)^t, (0, 1)^t$. So, $ImmL = R^2$

4.2 Matrix Representation

Theorem 4.2.1. *Let V, W be finite vector spaces with basis $E = [v_1, ..., v_n], F = [w_1, ..., w_m]$, respectively, and let $L : V \to W$ be a linear transformation. Then there exists an $m \times n$ matrix A called the matrix representation of L , with respect to the basis E, F, such that for any $v \in V$, $[L(v)]_F = A[v]_E$. Moreover, $A = ([L(v_1)]_F, [L(v_2)]_F, ..., [L(v_n)]_F)$*

Example 4.2.1. *Find the matrix representation of the following linear transformations*

(a) $L : P_2 \to R^2, L(f(x)) = \begin{pmatrix} \int_0^1 f(x)dx \\ f(0) \end{pmatrix}$ *with respect to the standard basis*

(b) $L : P_3 \to P_2, L(f(x)) = f'(x)$ *with respect to the standard basis*

(c) $L : P_3 \to P_2, L(f(x)) = f'(x)$ *with respect to the basis* $[1-x, 2x, x^2+x], [1, x]$

(d) $L : P_3 \to P_2, L(f(x)) = f'(x)$ *with respect to the basis* $[1-x, 2x, x^2+x], [x, 1]$

(e) $L : R^4 \to R^2, L((x_1, x_2, x_3, x_4)^t) = (x_1 - x_2 + x_3, x_4)^t$ *with respect to the standard basis*

Solution:

(a) $A = (L(1)_{[e_1,e_2]}, L(x)_{[e_1,e_2]}) = ((1,1)^t_{[e_1,e_2]}, (\frac{1}{2}, 0)^t_{[e_1,e_2]}) = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$.

(b) $A = (L(1)_{[1,x]}, L(x)_{[1,x]}, L(x^2)_{[1,x]}) = (0_{[1,x]}, 1_{[1,x]}, 2x_{[1,x]}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(c) $A = (L(1-x)_{[1,x]}, L(2x)_{[1,x]}, L(x^2+x)_{[1,x]}) =$
$(-1_{[1,x]}, 2_{[1,x]}, (2x+1)_{[1,x]}) =$
$\begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

(d) $A = (L(1-x)_{[x,1]}, L(2x)_{[x,1]}, L(x^2+x)_{[x,1]}) =$
$(-1_{[x,1]}, 2_{[x,1]}, (2x+1)_{[x,1]}) =$
$\begin{pmatrix} 0 & 0 & 2 \\ -1 & 2 & 1 \end{pmatrix}$.

(e) $A = (L(e_1)_{[e_1,e_2]}, L(e_2)_{[e_1,e_2]}, L(e_3)_{[e_1,e_2]}, L(e_4)_{[e_1,e_2]}) = ((1,0)^t_{[e_1,e_2]}, (-1,0)^t_{[e_1,e_2]}, (1,0)^t_{[e_1,e_2]}, (0,1)^t_{[e_1,e_2]}) = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

Chapter 5

Inner Products

Chapter 6

Eigenvalues

6.1 Definitions, Examples, and Basic Properties

Definition 6.1.1. *Let A be a square $n \times n$ matrix. A nonzero vector $\boldsymbol{x} \in R^n$ is called an eigenvector of A iff there exists a scalar $\lambda \in R$ such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$, and λ is called an the eigenvalue of A corresponding to the eigenvector $\boldsymbol{x}$.*

Now how to find the eigenvalues an eigenvectors of a square matrix A Let A be a square $n \times n$ matrix. Then the following are equivalent

(a) A nonzero eigenvector $\mathbf{x} \in R^n$ is an eigenvector of A corresponding to the eigenvalue $\lambda \in R$.

(b) $A\mathbf{x} = \lambda\mathbf{x}$

(c) $A\mathbf{x} - \lambda I_n\mathbf{x} = 0$

(d) $(A - \lambda I_n)\mathbf{x} = 0$

(e) The homogeneous system $(A - \lambda I_n)\mathbf{x} = 0$ ha a nonzero solution $\mathbf{x}$

(f) $N(A - \lambda I_n) \neq \{0\}$

(g) $(A - \lambda I_n)$ is singular

(h) $|A - \lambda I_n| = 0$

Definition 6.1.2. *Let A be a square $n \times n$ matrix. The equation $|A - \lambda I_n| = 0$ is called the characteristic equation of A, and the polynomial $p_A(\lambda) = |A - \lambda I_n|$ is called the characteristic polynomial of A.*

Theorem 6.1.3. *Let A be a square $n \times n$ matrix. Then the eigenvalues of A are the solutions of $|A - \lambda I_n| = 0$ and the corresponding eigenvectors of an eigenvalue λ is the solution of $(A - \lambda I_n)x = 0$, that is the eigenvalues are $N(A - \lambda I_n)x = 0$ and it is called the eigenspace of λ*

Example 6.1.1. *Find the eigenvalues and the corresponding eigenvector of* $A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$

Solution. We solve $|A - \lambda I| = 0$, so we get $\lambda = 1, \lambda = 2$
To find the corresponding eigenvectors, we solve the homogeneous system $(A - \lambda I_n)x = 0$

(a) For $\lambda = 1$, $(A - 1.I_n)x = 0$, so we solve $\begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix} x = 0$. We get $x = (1, 0)^t$ an eigenvector of A corresponding to the eigenvalue $\lambda = 1$.

(b) For $\lambda = 2$, $(A - 2.I_n)x = 0$, so we solve $\begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} x = 0$. We get $x = (3, 1)^t$ an eigenvector of A corresponding to the eigenvalue $\lambda = 2$.

Example 6.1.2. *Find the eigenvalues and the corresponding eigenvector of* $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

Solution. We solve $|A - \lambda I| = 0$, so we get $(1 - \lambda)(1 - \lambda) - 2 = 0$. So, $\lambda^2 - 2\lambda - 1 = 0$. By the quadratic formulae, we get $\lambda = 1 \pm \sqrt{2}$
To find the corresponding eigenvectors we solve the homogeneous system $(A - \lambda I_n)x = 0$

(a) For $\lambda = 1 + \sqrt{2}$, we solve $\begin{pmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{pmatrix} x = 0, \sqrt{2}R_2 + R_1 \Rightarrow$ $\begin{pmatrix} -\sqrt{2} & 2 \\ 0 & 0 \end{pmatrix}$. We get $x = (\sqrt{2}, 1)^t$ an eigenvector of A corresponding to the eigenvalue $\lambda = 1 + \sqrt{2}$.

(b) For $\lambda = 1 - \sqrt{2}$. We solve $\begin{pmatrix} \sqrt{2} & 2 \\ 1 & \sqrt{2} \end{pmatrix} x = 0$. We get $x = (-\sqrt{2}, 1)^t$ an eigenvector of A corresponding to the eigenvalue $\lambda = 1 - \sqrt{2}$.

Example 6.1.3. *Find the eigenvalues and the corresponding eigenvector of* $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Solution. We solve $|A - \lambda I| = 0$, so we get $(1-\lambda)(1-\lambda) - 1 = 0$. So, $\lambda^2 - 2\lambda = 0$. So, $\lambda = 0, 2$
To find the corresponding eigenvectors, we solve the homogeneous system $(A - \lambda I_n)x = 0$

(a) For 0, we solve $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} x = 0$. We get $x = (-1, 1)^t$ an eigenvector of A corresponding to the eigenvalue $\lambda = 0$.

(b) For 2. We solve $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} x = 0$. We get $x = (1, 1)^t$ an eigenvector of A corresponding to the eigenvalue $\lambda = 2$.

Similar matrices

Definition 6.1.4. *A square $n \times n$ matrices A, B are called similar matrices iff there exists a nonsingular matrix X such that $A = XBX^{-1}$.*

Theorem 6.1.5. *Let A, B be a square $n \times n$ similar matrices then A, B have the same eigenvalues.*

Proof. Enough to show A, B have the same characteristic polynomials. But $P_A(\lambda) = |A - \lambda I_n| = |XBX^{-1} - \lambda I_n| = |XBX^{-1} - \lambda XX^{-1}| = |X(B - \lambda I)X^{-1}| = |B - \lambda I_n| = P_B(\lambda)$. □

Remark 6.1.6. Let A, B be a square $n \times n$ similar matrices. Then

(a) $|A| = |B|$

(b) A, B have the same eigenvalues, but need not have the same eigenvectors.

Definition 6.1.7. *Let A be a square $n \times n$ matrix, the trace of A denoted by $tr(A)$ is the sum of the entries in the main diagonal.*

Theorem 6.1.8. *Let A be a square $n \times n$ matrix with eigenvalues $\lambda_1, ..., \lambda_n$. Then*

(a) $|A| = \lambda_1 .... \lambda_n$

(b) $tr(A) = \lambda_1 + ... + \lambda_n$

Theorem 6.1.9. *Let A be a square $n \times n$ matrix. Then A is singular iff 0 is an eigenvalue*

Theorem 6.1.10. *Let A be a square $n \times n$ matrix. Then A and A^t have the same eigenvalues*

Theorem 6.1.11. *Let A be an $n \times n$ matrix. If λ is an eigenvalue of A. If $n \in Z^+$, then λ^n is an eigenvalue of A^n with the same eigenvectors.*

Example 6.1.4. *Find the eigenvalues and the corresponding eigenvector of A^{100} if $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$*

Solution. From Example 6.1.3 above, the eigenvalues and the corresponding eigenvectors of A are: $\lambda = 0, 2$ with $x = (-1, 1)^t$ an eigenvector of A corresponding to the eigenvalue $\lambda = 0$, and $x = (1, 1)^t$ an eigenvector of A corresponding to the eigenvalue $\lambda = 2$. So the eigenvalues of A^{100} are $0^{100} = 0$ with $x = (-1, 1)^t$ an eigenvector, and 2^{100} with $x = (1, 1)^t$ an eigenvector. Also, $A^{100} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2^{100} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2^{100} \\ 2^{100} \end{pmatrix}$

Theorem 6.1.12. *Let A be a square nonsingular $n \times n$ matrix. If λ is an eigenvalue of A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with the same eigenvectors.*

Definition 6.1.13. *Let A be a square $n \times n$ matrix, and let λ be an eigenvalue of A. Then*

(a) The algebraic multiplicity of λ denoted by $alg(\lambda)$ is the number of how many λ is repeated.

(b) The geometric multiplicity of λ denoted by $gem(\lambda)$ is the number of li eigenvectors of λ, that is $gem(\lambda) = dimN(A - \lambda I)$

6.1.1 Complex eigenvalues

Recall that a complex number is of the from $x + yi$ where $x, y \in R, i^2 = -1$. If $z = a + bi$, then the conjugate of z denoted by $\overline{z} = a - bi$

Theorem 6.1.14. *(**Fundamental theorem of algebra**) Let $f(z) = c_n z^n + ... + c_1 z + c_0, c_i' s \in C$ be a complex polynomial. Then $f(z)$ has exactly n roots counting multiplicity.*

Theorem 6.1.15. *Let $f(z) = c_n z^n + ... + c_1 z + c_0, c_i' s \in R$ be a complex polynomial with real entries. If z_0 is a root of $f(z)$, then $\overline{z_0}$ is a root*

Theorem 6.1.16. *Let A be a square $n \times n$ matrix with real entries, and let λ be an eigenvalue of A with an eigenvector x. Then $\overline{\lambda}$ is an eigenvalue of A with eigenvector $\overline{x}$*

Example 6.1.5. *Find the eigenvalues and the corresponding eigenvector of $A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$*

Solution. We solve $|A - \lambda I| = 0$, so we get $(1 - \lambda)(-1 - \lambda) + 2 = 0$. So, $\lambda^2 + 1 = 0$. So, $\lambda = \mp i$
To find the corresponding eigenvectors, we solve the homogeneous system $(A - \lambda I_n)x = 0$

(a) For i, we solve $\begin{pmatrix} 1-i & -2 \\ 1 & -1-i \end{pmatrix} x = 0$, perform $(1-i)R_2 - R - 1$. We get $x = (1 + i, 1)^t$ an eigenvector of A corresponding to the eigenvalue $\lambda = i$.

(b) $x = (1 - i, 1)^t$ an eigenvector of A corresponding to the eigenvalue $\lambda = -i$. Do it similarly.

6.2 Diagonalization: section3 in the book

Definition 6.2.1. *A square $n \times n$ matrix A is called diagonalizable iff A is similar to a diagonal matrix D, that iff there exist a nonsingular matrix X, and a diagonal matrix D such that $A = XDX^{-1}$, and X is called the matrix that diagonalize A. A matrix that is not diagonalizable is called defective.*

Theorem 6.2.2. *The eigenvectors corresponding to distinct eigenvalues are li*

Theorem 6.2.3. *A square $n \times n$ matrix A is diagonalizable iff A has n li eigenvectors.*

Theorem 6.2.4. *If all the eigenvalues of a matrix A are distinct, then A is diagonalizable*

Remark 6.2.5. Let an $n \times n$ matrix A be diagonalizable. The proof of the above theorem gives a technique to find $X, D : A = XDX^{-1}$ as follows, let the eigenvalues of A be $\lambda_1, ..., \lambda_n$ counting multiplicity with corresponding eigenvectors $X_1, ..., X_n$. Take $X = (X_1, ..., X_n), D = diag(\lambda_1, ..., \lambda_n)$, then $A = XDX^{-1}$

Example 6.2.1. *Is $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ diagonalizable, if yes find X, D such that $A = XDX^{-1}$*

Solution. The eigenvalues of A are $\lambda = 0, 2$ which are distinct, so A is diagonalizable
To find the corresponding eigenvectors, we solve the homogeneous system $(A - \lambda I_n)x = 0$

(a) For 0, we get $x = (-1, 1)^t$ an eigenvector of A corresponding to the eigenvalue $\lambda = 0$.

(b) For 2, we get $x = (1, 1)^t$ an eigenvector of A corresponding to the eigenvalue $\lambda = 2$.

Let $X = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$